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CR- SUBMANIFOLDS OF A NEARLY TRANS-HYPERBOLIC SASAKIAN MANIFOLD WITH A QUARTER SYMMETRIC METRIC CONNECTION

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Abstract. We define a quarter symmetric metric connection in a nearly trans-hyperbolic sasakian manifold and we study CR- submanifolds of a nearly trans-hyperbolic sasakian manifold with a quarter symmetric metric connection. Moreover, we discuss the parallel distribution relating to ξ-vertical CR-submanifolds of a nearly trans-hyperbolic sasakian manifold with a quarter symmetric metric connection

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1. Introduction

A. Bejancu introduced the notion of CR-submanifolds of a Kaehler manifold in [5]. Latter, CR- submanifold have been studied by Kobayashi[1], Shahid et al. [18, 19], Yano and Kon [22] and others. Upadhyay and Dube [21] have studied almost contact hyperbolic (f,g,η,ξ) -structure, Dube and Mishra [9] have considered Hypersurfaces im-mersed in an almost hyperbolic Hermitian manifold also Dube and Niwas [10] worked with almost r-contact hyperbolic structure in a product manifold. Gherghe studied on harmonicity on nearly trans-Sasaki manifolds [11]. Bhatt and Dube [7] studied on CR-submanifolds of trans- hyperbolic Sasakian manifold. Joshi and Dube [14] studied on Semi-invariant submanifold of an almost r-contact hyperbolic metric manifold. Gill and Dube have also worked on CR submanifolds of trans-hyperbolic Sasakian manifolds [12].

Let ∇ be a linear connection in an n-dimensional differentiable manifold $\bar{M}$. The torsion tensor T and the curvature tensor R of ∇ are given respectively by [8]

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z$$

The connection ∇ is symmetric if the torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is metric if there is a Riemannian metric g in $\bar{M}$ such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. In [13], S. Golab introduced the idea of a quarter-symmetric connection. A linear connection is said to be a quarter-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

where η is a 1-form. In [2, 4], M. Ahmad et al. studied some properties of hypersurfaces of an almost r-paracontact Riemannian manifold with connections and also in [1, 3,15, 20] studied properties of CR-submanifolds of a nearly trans-Sasakian hyperbolic manifolds with connections.

2. Preliminaries

Let $\bar{M}$ be an n dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure (ϕ, ξ, η, g) where a tensor ϕ of type (1, 1), a vector field ξ, called structure vector field and η, the dual 1-form of is a 1-form ξ satisfying the following

$$\phi^2 X = X - \eta(X)\xi, \quad g(X, \xi) = \eta(X) \tag{1}$$

$$\phi(\xi) = 0, \quad \eta o\phi = 0, \quad \eta(\xi) = -1 \tag{2}$$

$$g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y), \tag{3}$$

for any X, Y tangents to $\bar{M}$ [4]. In this case

$$g(\phi X, Y) = -g(X, \phi Y) \tag{4}$$

An almost hyperbolic contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called trans-hyperbolic Sasakian [6] if and only if

$$(\bar{\nabla}_X \phi)Y = \alpha\{g(X, Y)\xi - \eta(Y)\phi X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\} \tag{5}$$

for all X, Y tangents to $\bar{M}$ and α, β are functions on $\bar{M}$. On a trans-hyperbolic Sasakian manifold M, we have

$$\bar{\nabla}_X \xi = -\alpha(\phi X) + \beta\{X - \eta(X)\xi\} \tag{6}$$

a Riemannian metric g and Riemannian connection $\bar{\nabla}$.

Further, an almost contact metric manifold $\bar{M}$ on (ϕ, ξ, η, g) is called nearly trans-hyperbolic Sasakian if [5]

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = \alpha\{2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi X\} \tag{7}$$

On other hand, a quarter symmetric metric connection $\bar{\nabla}$ on M is defined by

$$\bar{\nabla}_X Y = \bar{\nabla}^*_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi \tag{8}$$

Using (2.1), (2.2) and (2.6) in (2.5) and (2.6), we get respectively

$$\begin{aligned}(\bar{\nabla}_X \phi)Y &= \alpha\{g(X, Y)\xi - \eta(Y)\phi X\} \\ &\quad + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\} + g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi\end{aligned} \tag{9}$$

$$\bar{\nabla}_X \xi = -(\alpha + 1)\phi X + \beta\{X - \eta(X)\xi\} \tag{10}$$

In particular, an almost contact metric manifold $\bar{M}$ on (ϕ, ξ, η, g) is called nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a quarter symmetric metric connection if

$$\begin{aligned}(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X \\ = \alpha\{2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi X\} \\ -\eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X, Y)\xi\end{aligned} \tag{11}$$

Now, let M be a submanifold immersed in $\bar{M}$. The Riemannian metric induced on M is denoted by the same symbol g. Let TM and $T^{\perp}M$ be the Lie algebras of vector fields tangential to M and normal to M respectively and ∇ be the induced Levi-Civita connection on M, then the Gauss and

Weingarten formulas for the quarter symmetric metric connection are given by

$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y)$ (12)
$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N + \eta(N)\phi X$ (13)
for any $X, Y \epsilon TM$ and $V \epsilon T^{\perp}M$, where $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_N is the Weingarten map associated with N as
$g(A_N X, Y) = g(h(X,Y), N)$ (14)
For any $x \epsilon M$ and $X \epsilon T_X M$, we write
$X = PX + QX$ (15)
where $PX \epsilon D$ and $QX \epsilon D^{\perp}$.
Similarly for N normal to M, we have
$\phi N = BN + CN$ (16)
where BN (respectly CN) is the tangential component (respectly normal component) of ϕN.

Definition. An m dimensional Riemannian submanifold M of $\overline{M}$ is called a CR-submanifold of M if there exists a differentiable distribution$D : x \to D_X$ on M satisfying the following conditions:
(i) D is invariant, that is $\phi D_X \subset D_X$ for each $x \epsilon M$,
(ii) The complementary orthogonal distribution $D^{\perp}: X \to \mathrm{D}_X^{\perp} \subset T_X M$ of D is anti-invariant, that is, $\phi \mathrm{D}_X^{\perp} \subset \mathrm{T}_X^{\perp} M$ for each $x \epsilon M$. If $\dim \mathrm{D}_X^{\perp} = 0$ (respectly $\dim D_X = 0$), then the CR-submanifold is called an invariant (respectly, anti-invariant) submanifold. The distribution D (respectly, $D^{\perp}$) is called the horizontal (respectly, vertical) distribution. Also, the pair $(D, D^{\perp})$ is called ξ-horizontal (respectly, vertical) if $\xi_X \epsilon D_X$_(respectly, $\xi_X \epsilon \mathrm{D}_X^{\perp}$).

3. Some basic lemmas

Lemma 1. If M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter symmetric metric connection, then

$$P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - P\mathrm{A}_{\phi QY} X - P\mathrm{A}_{\phi QX} Y \tag{17}$$
$$= 2(\alpha+1) g(X,Y) P\xi - \alpha\eta(Y)\phi PX - \alpha\eta(X)\phi PY - \beta\eta(Y)\phi PX - \beta\eta(X)\phi PY$$
$$-\eta(X) PY - \eta(Y) PX + 4\eta(X)\eta(Y) P\xi + \phi P \nabla_X Y + \phi P \nabla_Y X$$

$$Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - Q\mathrm{A}_{\phi QY} X - Q\mathrm{A}_{\phi QX} Y \tag{18}$$
$$= 2Bh(X,Y) + 2(\alpha+1) g(X,Y) Q\xi - \alpha\eta(Y)\phi QX - \alpha\eta(X)\phi QY$$
$$-\eta(X) QY - \eta(Y) QX + 4\eta(X)\eta(Y) Q\xi$$

$$h(X, \phi PY) + h(Y, \phi PX) + \nabla_X^{\perp} \phi QY + \nabla_Y^{\perp} \phi QX \tag{19}$$
$$= \phi Q \nabla_Y X + \phi Q \nabla_X Y + 2Ch(X,Y) - \beta\eta(Y)\phi QX - \beta\eta(X)\phi QY$$

for any $X, Y \epsilon TM$.

Proof. Using (4), (9) and (10) in (11) we get

$$(\nabla_X \phi PY) + h(X, \phi PY) - \mathrm{A}_{\phi QY} X + \nabla_X^{\perp} \phi QY - \phi(\nabla_X Y) - \phi h(X,Y)$$
$$+(\nabla_Y \phi PX) + h(Y, \phi PX) - \mathrm{A}_{\phi QX} Y + \nabla_Y^{\perp} \phi QX - \phi(\nabla_Y X) - \phi h(Y,X)$$
$$= \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi X\}$$
$$-\eta(X) Y - \eta(Y) X + 4\eta(X)\eta(Y)\xi + 2g(X,Y)\xi$$

Again using (15) we get

$$P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - PA_{\phi QY}X - PA_{\phi QX}Y - \phi P\nabla_X Y \tag{20}$$
$$-\phi Q\nabla_X Y - \phi P\nabla_Y X - \phi Q\nabla_Y X + Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX)$$
$$-QA_{\phi QY}X - QA_{\phi QX}Y + h(X, \phi PY) + h(Y, \phi PX) + \nabla_X^{\perp} \phi QY$$
$$+\nabla_Y^{\perp} \phi QX - 2Bh(X,Y) - 2Ch(X,Y) = 2\alpha g(X,Y)P\xi$$
$$+2\alpha g(X,Y)Q\xi - \alpha\eta(Y)\phi PX - \alpha\eta(Y)\phi QX - \alpha\eta(X)\phi PY$$
$$-\alpha\eta(X)\phi QY - \beta\eta(Y)\phi PX - \beta\eta(Y)\phi QX - \beta\eta(X)\phi PY$$
$$-\beta\eta(X)\phi QY - \eta(X)PY - \eta(X)QY - \eta(Y)PX - \eta(Y)QX$$
$$+4\eta(X)\eta(Y)P\xi + 4\eta(X)\eta(Y)Q\xi + 2g(X,Y)P\xi + 2g(X,Y)Q\xi$$

for any $X, Y \epsilon TM$.

Now equating horizontal, vertical, and normal components in (20), we get the desired result.

Lemma 2. If M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a quarter symmetric metric connection, then

$$2(\bar{\nabla}_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X,Y] \tag{21}$$
$$+\alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi X\}$$
$$-\eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X,Y)\xi$$

$$2(\bar{\nabla}_Y \phi)X = \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi X\} \tag{22}$$
$$-\eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X,Y)\xi - \nabla_X \phi Y + \nabla_Y \phi X$$
$$-h(X, \phi Y) + h(Y, \phi X) + \phi[X,Y]$$

Proof. From Gauss formula (12), we have

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = \nabla_X \phi Y + h(X, \phi Y) - \nabla_Y \phi X - h(Y, \phi X) \tag{23}$$

Also we have

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = (\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X + \phi[X,Y] \tag{24}$$

From (22) and (23), we get

$$(\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X = \nabla_X \phi Y + h(X, \phi Y) - \nabla_Y \phi X - h(Y, \phi X) - \phi[X,Y] \tag{25}$$

Also for nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a quarter symmetric metric connection, we have

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X$$
$$= \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi X\}$$
$$-\eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X,Y)\xi \tag{26}$$

Adding (3.9) and (3.10), we get

$$2(\bar{\nabla}_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X,Y]$$
$$+\alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi X\}$$
$$-\eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X,Y)\xi$$

Subtracting (25) from (26) we get

$$2(\bar{\nabla}_Y \phi)X = \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi X\}$$
$$-\eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X,Y)\xi - \nabla_X \phi Y + \nabla_Y \phi X$$
$$-h(X, \phi Y) + h(Y, \phi X) + \phi[X,Y]$$

Hence Lemma is proved.

Lemma 3. If M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a quarter symmetric metric connection, then

$$\begin{aligned}2(\bar{\nabla}_Y\phi)(Z) &= \mathrm{A}_{\phi Y}Z - \mathrm{A}_{\phi Z}Y - \nabla_Z^{\perp}\phi Y + \nabla_Y^{\perp}\phi Z - \phi[Y,Z] \\ &\quad +\alpha\{2g(Y,Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y\} - \beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\} \\ &\quad -\eta(Y)Z - \eta(Z)Y + 4\eta(Y)\eta(Z)\xi + 2g(Y,Z)\xi\end{aligned}$$

$$\begin{aligned}2(\bar{\nabla}_Z\phi)Y &= \alpha\{2g(Y,Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y\} - \beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\} \\ &\quad -\eta(Y)Z - \eta(Z)Y + 4\eta(Y)\eta(Z)\xi + 2g(Y,Z)\xi - \mathrm{A}_{\phi Y}Z + \mathrm{A}_{\phi Z}Y \\ &\quad +\nabla_Z^{\perp}\phi Y - \nabla_Y^{\perp}\phi Z + \phi[Y,Z]\end{aligned}$$

for any $Y, Z \epsilon D^{\perp}$.

Proof. From Weingarten formula (13), we have

$$\bar{\nabla}_Z\phi Y - \bar{\nabla}_Y\phi Z = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^{\perp}\phi Z - \nabla_Z^{\perp}\phi Y \tag{27}$$

Also, we have

$$\bar{\nabla}_Z\phi Y - \bar{\nabla}_Y\phi Z = (\bar{\nabla}_Y\phi)Z - (\bar{\nabla}_Z\phi)Y + \phi[Y,Z] \tag{28}$$

From (27) and (28), we get

$$(\bar{\nabla}_Y\phi)Z - (\bar{\nabla}_Z\phi)Y = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^{\perp}\phi Z - \nabla_Z^{\perp}\phi - \phi[Y,Z] \tag{29}$$

Also for nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a quarter symmetric metric connection, we have

$$\begin{aligned}&(\bar{\nabla}_Y\phi)Z + (\bar{\nabla}_Z\phi)Y = \\ &\alpha\{2g(Y,Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y\} - \beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\} \\ &\quad -\eta(Y)Z - \eta(Z)Y + 4\eta(Y)\eta(Z)\xi + 2g(Y,Z)\xi\end{aligned} \tag{30}$$

Adding (29) and (30), we get

$$\begin{aligned}2(\bar{\nabla}_Y\phi)(Z) &= \mathrm{A}_{\phi Y}Z - \mathrm{A}_{\phi Z}Y - \nabla_Z^{\perp}\phi Y + \nabla_Y^{\perp}\phi Z - \phi[Y,Z] \\ &\quad +\alpha\{2g(Y,Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y\} - \beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\} \\ &\quad -\eta(Y)Z - \eta(Z)Y + 4\eta(Y)\eta(Z)\xi + 2g(Y,Z)\xi\end{aligned}$$

Subtracting (29) from (30) we get

$$\begin{aligned}2(\bar{\nabla}_Z\phi)Y &= \alpha\{2g(Y,Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y\} - \beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\} \\ &\quad -\eta(Y)Z - \eta(Z)Y + 4\eta(Y)\eta(Z)\xi + 2g(Y,Z)\xi - \mathrm{A}_{\phi Y}Z \\ &\quad +\mathrm{A}_{\phi Z}Y + \nabla_Z^{\perp}\phi Y - \nabla_Y^{\perp}\phi Z + \phi[Y,Z]\end{aligned}$$

This proves our assertions.

Lemma 4. If M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a quarter symmetric metric connection, then

$$\begin{aligned}2(\bar{\nabla}_X\phi)Y &= \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(Y)\phi X + \eta(X)\phi Y\} \\ &\quad -\eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X,Y)\xi - \mathrm{A}_{\phi Y}X + \nabla_X^{\perp}\phi Y \\ &\quad -\nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y]\end{aligned}$$

$$\begin{aligned}2(\bar{\nabla}_Y\phi)X &= \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(Y)\phi X + \eta(X)\phi Y\} \\ &\quad -\eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X,Y)\xi + \mathrm{A}_{\phi Y}X - \nabla_X^{\perp}\phi Y \\ &\quad +\nabla_Y\phi X + h(Y,\phi X) + \phi[X,Y]\end{aligned}$$

for any $X \epsilon D$and $Y \epsilon D^{\perp}$.

Proof. By using Gauss equation and Weingarten equation for $X \epsilon D$and $Y \epsilon D^{\perp}$ respectively we get

$$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y \phi X - h(Y,\phi X) \tag{31}$$

Also, we have

$$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = (\overline{\nabla}_X \phi)Y - (\overline{\nabla}_Y \phi)X + \phi[X,Y] \tag{32}$$

From (31) and (32), we get

$$(\overline{\nabla}_X \phi)Y - (\overline{\nabla}_Y \phi)X = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y \phi X - h(Y,\phi X) - \phi[X,Y] \tag{33}$$

Also for nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter symmetric metric connection, we have

$$(\overline{\nabla}_X \phi)Y + (\overline{\nabla}_Y \phi)X = \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi X\} \tag{34}$$
$$-\eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X,Y)\xi$$

Adding (33) and (34), we get

$$2(\overline{\nabla}_X \phi)Y = \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(Y)\phi X + \eta(X)\phi Y\}$$
$$-\eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X,Y)\xi - \mathrm{A}_{\phi Y}X + \nabla_X^{\perp}\phi Y$$
$$-\nabla_Y \phi X - h(Y,\phi X) - \phi[X,Y]$$

Subtracting (25) from (26) we get

$$2(\overline{\nabla}_Y \phi)X = \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(Y)\phi X + \eta(X)\phi Y\}$$
$$-\eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X,Y)\xi + \mathrm{A}_{\phi Y}X - \nabla_X^{\perp}\phi Y$$
$$+\nabla_Y \phi X + h(Y,\phi X) + \phi[X,Y]$$

Hence Lemma is proved.

4. Parallel distributions

Definition. The horizontal (respectly, vertical) distribution D (respectly, $D^{\perp}$) is said to be parallel [1] with respect to the connection on M if $\nabla_X Y \epsilon D$ (respectly, $\nabla_Z W \epsilon D^{\perp}$) for any vector field $X, Y \epsilon D$ (respectly, $W, Z \epsilon D^{\perp}$).

Proposition 1. If M be a ξ-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter symmetric metric connection and the horizontal distribution D is parallel, then

$$h(X,\phi Y) = h(Y,\phi X) \tag{35}$$

for all $X,\ Y \epsilon D$.

Proof. Using parallelism of horizontal distribution D, we have

$$\nabla_X \phi Y \epsilon D,\ \nabla_Y \phi X \epsilon D \quad \text{for any } X,\ Y \epsilon D. \tag{36}$$

Thus using the fact that $X = QY = 0$ for $Y \epsilon D$, (18) gives

$$Bh(X,Y) = g(X,Y)Q\xi \quad \text{for any } X,\ Y \epsilon D. \tag{37}$$

Also, since

$$\phi h(X,Y) = Bh(X,Y) + Ch(X,Y), \tag{38}$$

then

$$\phi h(X,Y) = g(X,Y)Q\xi + Ch(X,Y) \text{ for any } X,\ Y \epsilon D. \tag{39}$$

Next from (19), we have

$$h(X,\phi Y) + h(Y,\phi X) = 2Ch(X,Y) = 2\phi h(X,Y) - 2g(X,Y)Q\xi, \tag{40}$$

for any $X,\ Y \epsilon D$. Putting $X = \phi X \epsilon D$ in (40), we get

$$h(\phi X,\phi Y) + h(Y,\phi^2 X) = 2\phi h(\phi X,Y) - 2g(\phi X,Y)Q\xi \tag{41}$$

or

$$h(\phi X,\phi Y)-h(Y,X)=2\phi h(\phi X,Y)-2g(\phi X,Y)Q\xi \quad (42)$$

Similarly, putting $Y=\phi Y \epsilon D$ in (4.6), we get

$$h(\phi Y,\phi X)-h(X,Y)=2\phi h(X,\phi Y)-2g(X,\phi Y)Q\xi. \quad (43)$$

Hence from (42) and (43), we have

$$\phi h(X,\phi Y)-\phi h(Y,\phi X)=g(X,\phi Y)Q\xi-g(\phi X,Y)Q\xi \quad (44)$$

Operating ϕ on both sides of (4.10) and using $\phi\xi=0$, we get

$$h(X,\phi Y)=h(Y,\phi X) \quad (45)$$

for all $X,\ Y \epsilon D$.

Now, for the distribution $D^{\perp}$, we prove the following proposition.

Proposition 2. If M be a ξ-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian Manifold $\bar{M}$ with a quarter symmetric metric connection and the distribution $D^{\perp}$ is parallel with respect to the connection on M, then

$$A_{\phi Y}Z+A_{\phi Z}Y\ \epsilon\ D^{\perp} \quad \text{for any } Y,Z\ \epsilon\ D^{\perp}. \quad (46)$$

Proof. Let $,Z \epsilon D^{\perp}$, then using Gauss and Weingarten formula (2.10), we obtain

$$-A_{\phi Z}Y+\nabla_Y^{\perp}\phi Z-A_{\phi Y}Z+\nabla_Z^{\perp}\phi Y=\phi\nabla_Y Z+\phi h(Y,Z)+\phi\nabla_Z Y+\phi h(Z,Y)$$
$$+\alpha\{2g(Y,Z)\xi-\eta(Y)\phi Z-\eta(Z)\phi Y\}-\beta\{\eta(Y)\phi Z+\eta(Z)\phi Y\}$$
$$-\eta(Y)Z-\eta(Z)Y+4\eta(Y)\eta(Z)\xi+2g(Y,Z)\xi \quad (47)$$

for any $Y,Z\ \epsilon\ D^{\perp}$. Taking inner product with $X \epsilon D$ in (47), we get

$$g\big(A_{\phi Y}Z,X\big)+g\big(A_{\phi Z}Y,X\big)=g(\nabla_Y Z,\phi X)+g(\nabla_Z Y,\phi X) \quad (48)$$

But the distribution $D^{\perp}$ is parallel, then $\nabla_Y Z \epsilon D^{\perp}$ and $\nabla_Z Y \epsilon D^{\perp}$, for any $Y,Z\ \epsilon\ D^{\perp}$. Thus from (48) we have

$$g\big(A_{\phi Y}Z,X\big)+g\big(A_{\phi Z}Y,X\big)=0 \quad \text{or} \quad g\big(A_{\phi Y}Z+A_{\phi Z}Y,X\big)=0 \quad (49)$$

which is equivalent to

$A_{\phi Y}Z+A_{\phi Z}Y \epsilon D^{\perp}$ for any $Y,Z\ \epsilon\ D^{\perp}$

and this completes the proof.

Definition : A CR-submanifold M of a nearly trans-hyperbolic Sasakian Manifold $\bar{M}$ with a quarter symmetric metric connection is said to be totally geodesic if $h(X,Y\ )\ =\ 0$ for $X \epsilon D$ and $Y \epsilon D^{\perp}$.

It follows immediately that a CR-submanifold is mixed totally geodesic if and only if $A_N X \epsilon D$ for each $X \epsilon D$ and $N \epsilon T^{\perp}M$.

Let $X \epsilon D$ and $Y \epsilon \phi D^{\perp}$. For a mixed totally geodesic ξ-vertical CR-submanifold M of a nearly trans hyperbolic Sasakian Manifold $\bar{M}$ with a quarter symmetric metric connection then from (9), we have

$$(\bar{\nabla}_X\phi)N\ =\ 0$$

Since $\bar{\nabla}_X\phi N=(\bar{\nabla}_X\phi)N+\ \phi(\bar{\nabla}_X N)$ so that $\bar{\nabla}_X\phi N=\ \phi(\bar{\nabla}_X N)$.

Hence in view of (2.13), we get

$$\bar{\nabla}_X\phi N=-A_{\phi N}X+\nabla_X^{\perp}\phi N=-\phi A_N X+\phi\nabla_X^{\perp}N$$

As $A_N X \epsilon D,\ \ \phi A_N X \epsilon D$, so $\phi\nabla_X^{\perp}N=0$ if and only if $\bar{\nabla}_X\phi N\epsilon D$.

Thus we have the following proposition.

Proposition3. If M be a mixed totally geodesic ξ-vertical CR-submanifold of a nearly trans hyperbolic Sasakian Manifold $\bar{M}$ with a quarter symmetric metric connection, then the normal section $N \epsilon \phi D^{\perp}$ is D parallel if and only if $\nabla_X \phi N \epsilon D$ for all $X \epsilon D$.

5. Integrability conditions of distributions

Lemma 5.1. If M be a CR-submanifold of a nearly trans-hyperbolic Sasakian Manifold $\bar{M}$ with a quarter symmetric metric connection, then

$$\left(\bar{\nabla}_{\phi X}\phi\right)Y = 2(\alpha+1)g(\phi X, Y)\xi - (\alpha+\beta)\eta(Y)X + (\alpha+\beta)\eta(X)\eta(Y)\xi \qquad (50)$$
$$-\eta(Y)\phi X + \eta(X)\bar{\nabla}_Y\xi + \phi(\bar{\nabla}_Y\phi)(X) - \eta(\bar{\nabla}_Y X)\xi$$

for any $X, Y \epsilon TM$.

Proof. For nearly trans-hyperbolic Sasakian Manifold $\bar{M}$ with a quarter symmetric metric connection, we have

$$\left(\bar{\nabla}_{\phi X}\phi\right)Y = 2(\alpha+1)g(\phi X, Y)\xi - (\alpha+\beta)\eta(Y)X + (\alpha+\beta)\eta(X)\eta(Y)\xi - \eta(Y)\phi X \qquad (51)$$

and we have

$$(\bar{\nabla}_Y\phi)\phi X = \bar{\nabla}_Y\phi^2 X - \phi(\bar{\nabla}_Y\phi X) = \bar{\nabla}_Y\phi^2 X - \phi(\bar{\nabla}_Y\phi X) + \phi(\phi\bar{\nabla}_Y X) - \phi(\phi\bar{\nabla}_Y X)$$
$$= \bar{\nabla}_Y X - \eta(X)\bar{\nabla}_Y\xi - \phi(\bar{\nabla}_Y\phi X - \phi\bar{\nabla}_Y X) - \phi(\phi\bar{\nabla}_Y X) \qquad (52)$$
$$(\bar{\nabla}_Y\phi)\phi X = -\eta(X)\bar{\nabla}_Y\xi - \phi(\bar{\nabla}_Y\phi)(X) + \eta(\phi\bar{\nabla}_Y X)\xi$$

by (52) in (51), we have

$$\left(\bar{\nabla}_{\phi X}\phi\right)Y = 2(\alpha+1)g(\phi X, Y)\xi - (\alpha+\beta)\eta(Y)X + (\alpha+\beta)\eta(X)\eta(Y)\xi \qquad (53)$$
$$-\eta(Y)\phi X + \eta(X)\bar{\nabla}_Y\xi + \phi(\bar{\nabla}_Y\phi)(X) - \eta(\bar{\nabla}_Y X)\xi$$

for any $X, Y \epsilon TM$, which completes the proof of the lemma. On a nearly trans-hyperbolic Sasakian Manifold $\bar{M}$ with a quarter symmetric metric connection, Nijenhuis tensor is given by

$$N_\phi(X,Y) = \left(\bar{\nabla}_{\phi X}\phi\right)Y - \left(\bar{\nabla}_{\phi Y}\phi\right)X - \phi(\bar{\nabla}_X\phi)Y + \phi(\bar{\nabla}_Y\phi)X \qquad (54)$$

for any $X, Y \epsilon TM$.

As of (50) and (54), we have

$$N_\phi(X,Y) = 4(\alpha+1)g(\phi X, Y)\xi + (\alpha+\beta)\eta(Y)X + 3(\alpha+\beta)\eta(X)Y + \eta(Y)\phi X$$
$$+3\eta(X)\phi Y - 4(\alpha+\beta)\eta(X)\eta(Y)\xi + \eta(X)\bar{\nabla}_Y\xi - \eta(Y)\bar{\nabla}_X\xi - \eta(\bar{\nabla}_Y X)\xi$$
$$+\eta(\bar{\nabla}_X Y)\xi + 4\phi(\bar{\nabla}_Y\phi)X$$

Proposition 5.2. If M be a CR-submanifold of a nearly trans-hyperbolic Sasakian Manifold $\bar{M}$ with a quarter symmetric metric connection, then

$$\left(A_{\phi Y}Z - A_{\phi Z}Y\right) = \phi P[Y,Z] + (2\alpha+3)[\eta(Y)Z - \eta(Z)Y]$$
$$+(\alpha+\beta)[\eta(Z)\phi PY - \eta(Y)\phi Z] - 2(\alpha+\beta)g(\phi PZ, Y)P\xi \qquad (55)$$

for any $Y, Z \epsilon D^{\perp}$.

Proof: For $Y, Z \epsilon D^{\perp}$ and $X \epsilon T(M)$, we have

$$2g\left(A_{\phi Z}Y,\ X\right) = 2g(h(X,Y), \phi Z) = g(h(X,Y), \phi Z) + g(h(X,Y), \phi Z)$$
$$= g(\bar{\nabla}_X Y, \phi Z) + g(\bar{\nabla}_Y X, \phi Z) = g(\bar{\nabla}_X Y + \bar{\nabla}_Y X, \phi Z) \qquad (56)$$

$= g(\phi(\overline{\nabla}_X Y + \overline{\nabla}_Y X), Z) = -g(\overline{\nabla}_X \phi Y + \overline{\nabla}_Y \phi X - (\overline{\nabla}_X \phi) Y - (\overline{\nabla}_Y \phi) X, Z)$

$= -g(\overline{\nabla}_X \phi Y, Z) - g(\overline{\nabla}_Y \phi X, Z) + g(\alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\}$

$-\beta\{\eta(X)\phi Y + \eta(Y)\phi X\} - \eta(X)Y - \eta(Y)X + 4\eta(X)\eta(Y)\xi + 2g(X,Y)\xi, Z)$

$= g\big(A_{\phi Y}Z, X\big) - g(\phi(\overline{\nabla}_Y Z), X) + 2\alpha g(\eta(Z)Y, X) + \alpha g(\eta(Y)\phi Z, X)$

$-\alpha g(g(\phi Y, Z)\xi, X) - \beta g(g(\phi Y, Z)\xi, X) + \beta g(\eta(Y)\phi Z, X)$

$-g(g(Y,Z)\xi, X) - g(\eta(Y)Z, X) + 4g(\eta(Y)\eta(Z)\xi, X) + 2g(\eta(Z)Y, X)$

The above equation is true for all $X \epsilon T(M)$, therefore transvecting the vector field X both sides, we have

$$A_{\phi Z}Y = A_{\phi Y}Z - \phi\overline{\nabla}_Y Z + 2\alpha\eta(Z)Y + \alpha\eta(Y)\phi Z - \alpha g(\phi Y, Z)\xi$$
$$-\beta g(\phi Y, Z)\xi + \beta\eta(Y)\phi Z - g(Y,Z)\xi - \eta(Y)Z + 4\eta(Y)\eta(Z)\xi + 2\eta(Z)Y \quad (57)$$

for any $Y, Z \epsilon D^{\perp}$. Interchanging the vector fields Y and Z, we get

$$2A_{\phi Y}Z = A_{\phi Z}Y - \phi\overline{\nabla}_Z Y + 2\alpha\eta(Y)Z + \alpha\eta(Z)\phi Y - \alpha g(\phi Z, Y)\xi \quad (58)$$
$$-\beta g(\phi Z, Y)\xi + \beta\eta(Z)\phi Y - g(Z,Y)\xi - \eta(Z)Y + 4\eta(Y)\eta(Z)\xi + 2\eta(Y)Z$$

Subtracting (57) and (58), we get

$$3\big(A_{\phi Y}Z - A_{\phi Z}Y\big) = \phi P[Y,Z] + (2\alpha + 3)[\eta(Y)Z - \eta(Z)Y] \quad (59)$$
$$+(\alpha + \beta)[\eta(Z)\phi PY - \eta(Y)\phi Z] - 2(\alpha + \beta)g(\phi PZ, Y)P\xi$$

for any $Y, Z \epsilon D^{\perp}$.

Theorem 5.1. If M be a CR-submanifold of a nearly trans-hyperbolic Sasakian Manifold $\overline{M}$ with a quarter symmetric metric connection, then the distribution $D^{\perp}$ is integrable if and only if

$$3\big(A_{\phi Y}Z - A_{\phi Z}Y\big) = \left(\frac{2\alpha+3}{3}\right)[\eta(Y)Z - \eta(Z)Y] \quad (60)$$

Proof: Primary suppose that the distribution $D^{\perp}$ is integrable. Then $[Y,Z] \epsilon D$ for any $Y, Z \epsilon D^{\perp}$. Since P is a projection operator on D, so $P[Y,Z] = 0$. Thus from (55) we get (60). Conversely, we suppose that (60) holds. Then using (55), we have $\phi P[Y,Z] = 0$ for any $Y, Z \epsilon D^{\perp}$. Since $rank\ \phi = 2n$. Therefore, either $P[Y,Z] = 0$ or $P[Y,Z] = k\xi$. But $P[Y,Z] = k\xi$ is not possible as P is a projection operator on D. Thus, $P[Y,Z] = 0$, which is equivalent to $[Y,Z] \epsilon D^{\perp}$ for any $Y, Z \epsilon D^{\perp}$ and hence $D^{\perp}$ is integrable.

Corollary 5.1. If M be a ξ-horizontal CR-submanifold of a nearly trans-hyperbolic Sasakian Manifold $\overline{M}$ with a quarter symmetric metric connection, then the distribution $D^{\perp}$ is integrable if and only if

$$A_{\phi Y}Z - A_{\phi Z}Y = 0 \quad (61)$$

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